Journal of Science-FAS-SEUSL (2025) 06(02) 01-10
ISSN:2738-2184

Study on Semi-Analytical Solutions of Fractional Derivatives with Applications

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Abstract

Differential equations are very useful tools to formulate real-world phenomena into mathematical form. In recent years, fractional calculus has gained considerable attention in modeling diverse real-world phenomena in science and engineering fields due to its memory effects and hereditary properties. In this study, we aim to review the ideas of fractional derivatives and their applications in fluid dynamics studies. The Adomian's Decomposition Method (ADM) has used to semi-analytically solve the Riemann-Liouville non-linear fractional-order differential equations. The ADM has the iteration steps that lead to highly accurate results. The study can also be used for numerous model examples of fractional derivatives with initial and boundary conditions involve. Among the many interesting applications of fractional derivatives are complex viscoelastic media, electrical spectroscopy, porous media, cosmology, environmental science, medicine (the modeling of infectious diseases), and signal and image processing. When we consider the negative side of fractional-based models, they are computationally expensive, require enormous data storage requirements, and have a phenomenological description that does not necessarily reflect the physical mechanism behind the scenes.

Keywords: Adomian's Decomposition Method (ADM), Fractional derivatives, Semi-analytical solution techniques.

1. Introduction

Real-world problems can be formulated into mathematical form through the use of differential equations. The fractional calculus is getting considerable attention and has gained a special interest in modeling diverse real-world phenomena in science and engineering fields due to its memory effects and hereditary properties. A fractional derivative is the derivative of any arbitrary order, real or complex.

The memory means an existence of output (response, endogenous variable) at the present time that depends on the history of the change of the input (impact, exogenous variable) on a finite (or infinite) time interval. The memory can be described by the function that is called the memory function, which is a kernel of the integral-differential operator (Vasily E. Tarasova & Svetlana S. Tarasova, 2019). The hereditary property of the medium, i.e., the dependence of the current state of a dynamic system on its previous. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property.

The ordinary classical models of integer-order derivatives are sometimes unable to model the proper complex dynamics applications. Since the integer-order differential operator is a local operator, whereas fractional-order derivatives involve non-local operator. This means that the next state of a system depends not only upon its current state but also upon all of its historical

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states. This is more realistic and it is one reason why fractional calculus has become more and more popular. This leads the fractional derivatives to be very convenient to describe such cases in chemistry, biology, mechanics, signal and image processing, and many more applications. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property.

There is a huge range of fractional models in the literature, such as fractional Brownian motion, fractional neutron point kinetic model, power law, Riesz potential, dynamical systems, mechanical systems, computational fractional derivative equations, fractional filters, fractional transforms, fractional wavelets, geophysics, and fluid dynamics, etc.

The most widely accepted definition including the singular kernel was proposed by Riemann and Liouville–Caputo. The Caputo fractional derivative definition has appeared frequently in the porous media literature.

Furthermore, fractional-order boundary value problems of linear and nonlinear fractional differential equations have been extensively investigated by many authors. By applying various techniques, many researchers have studied the existence of solutions of linear and nonlinear fractional derivatives.

There are several different analytical and numerical methods have been proposed to solve fractional ordinary differential equations. Including, Linear fractional differential equations by Operational Methods, (Roberto Garra, 2012), Classical Laplace Transform, (Rogosin S. & Dubatovskaya M., 2018) and Adomian's Decomposition Method, (V. Daftardar-Geji and H. Jafari, 2005), Variation Iteration Method, and Adams-Bashforth-Moulton Method and many more.

The Adomian's Decomposition Method (ADM) has been shown to solve effectively, easily, and accurately a large class of linear and nonlinear problems, generally one or two iterations lead to high accurate solutions. This method provides the solution in a rapidly convergent series with components that are elegantly computed. The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients as the method does not need linearization, weak nonlinearity assumptions or perturbation theory. Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution.

In this paper, we aim to analyze the study regarding the linear and nonlinear fractional derivatives and its application in real-world phenomena of science and engineering fields. Also, highlight their advantages over conventional continuum flow models toward reservoir simulation and management. Then, we tend to use the semi-analytic solving method of Adomian's Decomposition Method to derive the solutions for the test examples chosen. Also further establish and extend the usage of the fractional derivative applications using experimental studies.

2. Background of Study

The work of fractional calculus is carried out in a large number of articles. Including (Rogosin S. & Dubatovskaya M., 2018), present the essence of two important approaches in the fractional calculus, namely, those developed by Letnikov and Marchaud. The authors collect the most important results for the corresponding fractional derivatives, compare these constructions and its applications.

(Farnaz, Mubashir Qayyum, Syed Inayat Ali Shah, & Sha, 2021), analyzed in fractional space on Oldroyd-6 fluid in thin film flow context for both lifting and drainage scenarios. The posed highly nonlinear boundary value ordinary differential equations of the problem are converted into fractional differential equations.

(Anupama Choudhary, Devendra Kumar, & Jagdev Singh, 2016), discussed a fractional model arising in flow of two incompatible liquids through homogenous porous media with mean capillary pressure. The solution is derived by the application of the Sumudu transform and the Fourier sine transform.

(Umair Ali, 2012), suggested fractional functional for the Adomian's Decomposition Method (ADM) to solve the nonlinear fractional order partial differential equations with fractional order initial and boundary conditions by using the modified Riemann–Liouville fractional derivative proposed by G. Jumarie. The proposed modification in Adomian's Decomposition Method has made the results more powerful and efficient than before in finding analytical as well as numerical solutions for a wide class of linear and nonlinear fractional differential equations.

From among many advanced numerical methods applied to calculation, one of the widely applied techniques are perturbation methods. In 1999, Ji Huan, has proposed a new perturbation technique coupled with the Homotopy technique, which is called the Homotopy Perturbation Method (HPM), (He J.H, 1999). The HPM has attracted the attention of researchers in recent years with many advantages is the fact that it produces the approximate solution quite fast and results are more reliable comparing other numerical methods.

Furthermore, the Homotopy Perturbation Method (HPM), which gives the approximate solutions for nonlinear fractional differential equations and also used as numerical solution for multi-order fractional derivatives. This method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method does not require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences.

(Z.Z. Ganji, 2008), applied the homotopy perturbation method (HPM) to fractional differential equations and revealed that the HPM is an alternative analytical method for fractional differential equations. (Odibat., 2007), presented the modified homotopy perturbation method for calculating approximate solutions for nonlinear partial differential equations of fractional order. The results obtained using this method gave very high accuracy comparing with the variation of iteration method. (Rizwan Ali , Muhammad Imran Asjad, & Ali Akgül , 2021) , have analyzed the mathematical fractional model of hybrid viscous nanofluids and its application in heat and mass transfer. Which explains the application of novel way of modeling of heat and mass transfer flow

of hybrid nanofluid (Aluminum and Copper) for different base fluid water and engine oil. The governing equations for energy and momentum equations are developed with Caputo fractional power law derivative through constitutive relations. The flow of nanofluids confined between the two parallel plates with distanced apart. This model can be solved by means of the Laplace transform technique. Statically analysis for Nusselt number and Sherwood number is also discussed.

The gas transport in heterogeneous media has been studied by (HongGuang Suna, Yong Zhang, Dumitru Baleanu, Wen Chen, & YangQuan Chen, 2018). The gas transport in heterogeneous media has an important influence on oil-gas exploitation and development. Therefore, injecting gas into oil or gas reservoirs can significantly reduce oil viscosity, mitigate atmospheric emissions and control climate change to enhance oil or gas recovery efficiency and protect the environment. However, it is well-known that the random motion in gas transport in natural reservoirs deviates from the normal Brownian motion whose scaling limit cannot be properly described by classical models (such as Darcy's law and the advection dispersion equation) due to heterogeneity and complexity of the medium structure.

Anomalous transport of gas exhibits obvious path and history dependent behaviors. Subsequently, the fractional derivative models have been applied to explain the time memory and space non-locality in gas transport. Applicability of the fractional derivative models have been efficiently verified by employing a set of experimental data in the literature which are compared well with the results of numerical simulation and analytical solution. (Ailian Chang, HongGuang Sun, Yong Zhang, & Chunmiao , 2018).

3. Application of Fractional derivatives in Fluid flow

Recently, great interest has been devoted to the application of fractional derivative modelling to different fields of science and engineering. Among the many interesting applications of fractional calculus to physical systems, complex viscoelastic media, electrical spectroscopy, porous media, cosmology, environmental science, signal and image processing, medicine (the modelling of infectious diseases), and materials, etc.

In fluid mechanics, fluid flow through porous media is the manner in which fluids behave when flowing through a porous media, for example sponge or wood, or when filtering water using sand or another porous material. Classical flow mechanics in porous media assumes that the medium is homogenous, isotropic, and has an intergranular pore structure.

Further, many research articles demonstrating the application of the fractional derivative concept in reservoir engineering discipline as well. For example, there are studies related to modelling the long-tailed non Fickian solute transport in fractured porous rock, fluid flow in naturally fractured nano-porous media, anomalous heat transport in heterogeneous media, modelling fluid flow in thermally altered zones in porous media and many more (Caputo, 1967). Accurate modelling of heat transport in fractured porous media is essential for the applications of geothermal reservoirs, thermal enhanced oil recovery and reservoir management, that is, detection of water influx in the wellbore.

Reservoir Engineering is a branch of petroleum engineering that applies scientific principles to the fluid flow through porous medium during the development and production of oil and gas

reservoirs so as to obtain a high economic recovery. The working tools of the reservoir engineer are subsurface geology, applied mathematics, and the basic laws of physics and chemistry governing the behavior of liquid and vapor phases of crude oil, natural gas, and water in reservoir rock.

A detailed description of the pore network and the adequate description of the rheological behavior of the complex reservoir fluid(s) are fundamental to the accuracy of any fluid flow model developed to describe flow through reservoir rocks. (Hossain et al, 2008) attempted to address the issue of incorporating the time dependency of rock and fluid properties explicitly into the existing fluid flow models. The authors introduced a memory model to capture the time evolution of rock and fluid properties with time employing the fractional derivative concept. Furthermore, the parameters of the resulting fractional derivative models have clear physical significance and must be obtained from a data fitting of experimental or field measurements.

Also, when we consider the negative side of the fractional-based models, which are computationally expensive and require enormous data storage requirement and are of a phenomenological description which does not necessarily reflect the physical mechanism behind the scenes.

4. Methodology

Fractional derivatives are a branch of mathematical analysis that studies the several different possibilities of defining real number or complex number orders of derivatives. Historically, the concept of fractional derivatives can be traced back to Leibniz's letter to L'Hopital in 1695, where the derivative of a non-integer order (1/2) was mentioned. It can also be applicable in the field of fluid dynamics, where fractional-order models are used to capture memory effects and anomalous transport phenomena. In methodology, fractional derivatives are introduced through definitions such as Riemann–Liouville, Caputo, or Grünwald–Letnikov, which provide the mathematical framework for applying these concepts to physical systems.

4.1 Basic definitions of the fractional derivatives:

4.1.1 The Riemann – Liouville derivative of fractional order *α* of function *x*(*t*):

$$ {}_aD_x^{\alpha} f(x) = \frac{d^n}{dx^n}\, {}_aI_x^{n-\alpha} f(x) = \frac{d^n}{dt^n}\, \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt \ , \qquad ------ \ (1) $$

where the Riemann – Liouville operator is given by

$$ {}_aI_x^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt $$

and $n-1 \leq \alpha \leq n \in \mathbb{Z}^+$.

4.1.2 The Caputo derivative of fractional order α of function x(t):

$$ {}_aD_x^{\alpha} f(x) = D_x^{-(n-\alpha)} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^m(t)\, dt \,, \quad ------ (2) $$

From this definition, one can see that ${}_aD_x^{\alpha} f^{(m)}(x) = {}_aD_x^{\alpha+m} f(x)$, and the gamma function

$$ \Gamma = \int_0^{\infty} t^{z-1} e^{-t}\, dx $$

.

4.2 Solutions Methods of the Fractional Derivatives: Adomian's Decomposition Method:

The Adomian's Decomposition Method (ADM) provides the solution in a rapidly convergent series with components that are elegantly computed. The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients as the method does not need linearization, weak nonlinearity assumptions or perturbation theory.
Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution.

The basic idea of the Adomian's decomposition method is to decompose the unknown equation

$$ F(y) = g(t) \quad ------ (3) $$

First, the whole equation is decomposed into several parts as, $F = L + R + N$

Therefore, $F(y) = L(y) + R(y) + N(y) = g(t).$

That is, $L(y) = g(t) - R(y) - N(y)$.

Where L – Highest order derivative and easily invertible
R – Linear differential operator of order less than L
g – Source term
N – Nonlinear term satisfies Lipschitz condition.

The function $y(t)$ is assumed to be bounded for all $t \in I = [0,1]$. Choose suitable operator L such that L^{-1} exists, then operator L^{-1} is used to act on both sides of this equation, obtaining

$$ y = L^{-1} g - L^{-1} R(y) - L^{-1} N(y) + \Phi \quad , \quad ------- (4) $$

where, the term of Φ satisfies $L(\Phi) = 0$.

If we decompose the true solution y of the equation into the sum of infinity components of different degree solutions. That is,

$$ y = \sum_{n=0}^{\infty} y_n $$

The nonlinear term $N(y)$ is decomposed into the sum of infinite number Adomian's polynomials of $A_n(y_0, y_1, \ldots, y_n)$.

$$ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i y_i\right) \quad at\ \lambda = 0 \quad ------(5) $$

Then the equation (4) can be written as

$$\sum_{n=0}^{\infty} y_n = L^{-1}\, g - L^{-1}\, R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) + \Phi \qquad ------(6)$$

Now the Adomian's decomposition satisfies the following recurrence relation:

$$\begin{aligned} y_0 &= L^{-1}\, g + \Phi \\ y_1 &= -L^{-1}\, R(y_0) - L^{-1}\, A_0 \\ y_2 &= -L^{-1}\, R(y_1) - L^{-1}\, A_1 \\ y_{m+1} &= -L^{-1}\, R(y_m) - L^{-1}\, A_m \qquad ------(7) \end{aligned}$$

5. Results and Discussion

This section presents an example of applying the Adomian's decomposition method to solve fractional non-linear ordinary differential equations.

5.1 Model Example of the non-linear Fractional Derivative:

1. $\dfrac{dy}{dt} + \dfrac{d^{\frac{1}{2}}y}{dt^{\frac{1}{2}}} - 2y^2 = 0 \qquad ------(i)$

Solution:

Consider the linear operator $L = \frac{d}{dx} \quad \Rightarrow \quad L^{-1} = \int .\, dt$

Non-linear term $N(y) = y^2$

Now the equation (i) can be written as $L(y) = -\dfrac{d^{\frac{1}{2}}y}{dt^{\frac{1}{2}}} + 2y^2$ which implies

$$L^{-1} \Rightarrow \quad y = C - L^{-1}\left(\frac{d^{\frac{1}{2}}y}{dt^{\frac{1}{2}}}\right) + 2\, L^{-1}(y^2) \qquad ------(ii)$$

where C is an arbitrary integral constant.

Since L is a linear operator, we get

$$\begin{aligned} y &= C - \frac{d^{-\frac{1}{2}}y}{dt^{-\frac{1}{2}}} + 2\, L^{-1}(y^2) \\ &= C - I^{\frac{1}{2}}(y) + 2\, L^{-1}(y^2) \quad ------(ii) \end{aligned}$$

Where I is the Riemann-Liouville operator, and

$$I^{\alpha}(t^n) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\, t^{n+\alpha}\,.$$

Then the Adomian's decomposition solution is given by

$$y = \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + \cdots$$

For the non-linear form is

$$N(y) = \sum_{n=0}^{\infty} A_n \quad ; \quad A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i y_i\right) \quad at\ \lambda = 0\,. \quad ------(iii)$$

So, the equation (ii) is written as

$$\sum_{n=0}^{\infty} y_n = C - I^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} y_n\right) + 2\,L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \qquad ----------(iv)$$

Equating both-sides in the equation (iv) we get

$$y_0 = C \quad ,$$

$$y_{n+1} = -I^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} y_n\right) + 2\,L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \quad , \ for\ n \geq 0\,.$$

<u>For $n = 0$</u>: $\boldsymbol{y_1 = -I^{\frac{1}{2}}(y_0) + 2\,L^{-1}(A_0)}$ $\qquad ------(v)$

Using the equation (iii) we have $N(y) = y^2$, therefore

$$A_0 = N(y_0) = {y_0}^2 = C^2\,.$$

Then the equation (v) gives, when substituting $\alpha = \frac{1}{2}$

$$y_1 = -I^{\frac{1}{2}}(C) + 2\int C^2\,dt = -I^{\frac{1}{2}}(Ct^0) + 2\int C^2\,dt = C\left[\frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}\,t^{\frac{1}{2}}\right] + 2\,C^2 t$$

$$= -2C\,\frac{\sqrt{t}}{\sqrt{\pi}} + 2\,C^2 t\,. \qquad \left(\because\ \Gamma(1) = 0! = 1 \ \ and\ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}\right)$$

Using the equation (iii) we have $N(y) = y^2$, therefore

$$A_0 = N(y_0) = {y_0}^2 = C^2\,.$$

Then the equation (v) gives, when substituting $\alpha = \frac{1}{2}$

$$y_1 = -I^{\frac{1}{2}}(C) + 2\int C^2\,dt = -I^{\frac{1}{2}}(Ct^0) + 2\int C^2\,dt = C\left[\frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}\,t^{\frac{1}{2}}\right] + 2\,C^2 t$$

$$= -2C\,\frac{\sqrt{t}}{\sqrt{\pi}} + 2\,C^2 t\,. \qquad \left(\because\ \Gamma(1) = 0! = 1 \ \ and\ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}\right)$$

$$---(vi)$$

<u>For $n = 1$</u>: $\boldsymbol{y_2 = -I^{\frac{1}{2}}(y_1) + 2\,L^{-1}(A_1)}$ $\qquad ------(vii)$

Consider the Riemann-Liouville operator

$$I^{\frac{1}{2}}(y_1) = I^{\frac{1}{2}}\left(-2C\,\frac{\sqrt{t}}{\sqrt{\pi}} + 2\,C^2 t\right) = -\frac{2C}{\sqrt{\pi}} I^{\frac{1}{2}}\left(t^{\frac{1}{2}}\right) + 2\,C^2\,I^{\frac{1}{2}}(t)$$

Again substituting $\alpha = \frac{1}{2}$ and $n = \frac{1}{2}, 1$ we get

$$I^{\frac{1}{2}}(y_1) = -\frac{2C}{\sqrt{\pi}}\left[\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\,t\right] + 2\,C^2\left[\frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)}\,t^{\frac{3}{2}}\right] = -Ct + \frac{8C^2}{3\sqrt{\pi}}\,t^{\frac{3}{2}}$$

Now, for the non-linear form, we get

$$A_1 = \frac{d}{d\lambda} N(y_0 + \lambda y_1) = N'(y_0 + \lambda y_1)\,.\,y_1 \quad at\ \lambda = 0$$

$$= N'(y_0)\,.\,y_1 = 2y_0y_1$$

$$L^{-1}(A_0) = L^{-1}(2y_0y_1) = \int 2C\left[-2C\,\frac{\sqrt{t}}{\sqrt{\pi}} + 2\,C^2t\right] dt = -\frac{8C^2}{3\sqrt{\pi}}\,t^{\frac{3}{2}} + 2C^3t^2\,.$$

So, substituting all these results in the equation (vii), we can get

$$y_2 = -\left[-Ct + \frac{8C^2}{3\sqrt{\pi}}\,t^{\frac{3}{2}}\right] + 2\left[-\frac{8C^2}{3\sqrt{\pi}}\,t^{\frac{3}{2}} + 2C^3t^2\right]$$

$$= Ct - \frac{8C^2}{3\sqrt{\pi}}\,t^{\frac{3}{2}} + 4C^3t^2 \qquad ------(viii)$$

Therefore, the Adomian's decomposition solution is given by

$$y = \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + \cdots$$

$$= C + \left\{-2C\,\frac{\sqrt{t}}{\sqrt{\pi}} + 2\,C^2t\right\} + \left\{Ct - \frac{8C^2}{3\sqrt{\pi}}\,t^{\frac{3}{2}} + 4C^3t^2\right\} + \cdots$$

$$= C - 2C\,\frac{t^{\frac{1}{2}}}{\sqrt{\pi}} + (2\,C^2 + C)t - \frac{8C^2}{3\sqrt{\pi}}\,t^{\frac{3}{2}} + 4C^3t^2 + \cdots.$$

In a similar way, we can solve the linear non-linear ordinary and partial fractional differential equations and the system of fractional equations by Adomian's decomposition method.

6. Conclusion

In this research study, we have analyzed the ideas of fractional derivatives and their application in fluid dynamics studies. Furthermore, we have also derived the semi-analytical solution of the non-linear differential equations of fractional order using Adomian's decomposition method. The main finding of our study is that the method of Adomian's decomposition is very accurate, simple, and can be applied to solve linear and non-linear ordinary and partial differential equations of fractional order. It is also easier to determine the terms of the fractional differential equation approximation solution. When we consider the negative side of fractional-based models, they are computationally expensive, require enormous data storage requirements, and have a phenomenological description that does not necessarily reflect the physical mechanism behind the scenes.

Furthermore, the study can also use any simulation techniques of mathematical software to easily find the solutions and to represent them graphically, which is our future work.

Acknowledgment

The authors thank South Eastern University of Sri Lanka, for providing an environment and facilities to fulfill the research goal.

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